Section 3.5

Higher Derivatives

(1) Lagrange and Leibniz Notation
(2) Detecting Increase/Decrease
(3) Concavity
(4) Higher Derivatives and Implicit Differentiation

Notation for Higher Derivatives

Lagrange: $f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x),\ \ldots$

Leibniz: $\dfrac{dy}{dx},\ \dfrac{d^2y}{dx^2},\ \dfrac{d^3y}{dx^3},\ \ldots,\ \dfrac{d^ny}{dx^n},\ \ldots$

Example: If $f(x) = x^2 + 3\ln(x)$, then

$$f'(x) = \frac{dy}{dx} = 2x + 3x^{-1} \qquad f'''(x) = \frac{d^3y}{dx^3} = 6x^{-3}$$

$$f''(x) = \frac{d^2y}{dx^2} = 2 - 3x^{-2} \qquad f^{(4)}(x) = \frac{d^4y}{dx^4} = -18x^{-4}$$

Higher Derivatives of Polynomial Functions

$$
\begin{aligned}
h(t) &= -4.9t^2 + v_0 t + h_0 \\
h'(t) &= -9.8t + v_0 \\
h''(t) &= -9.8 \\
h'''(t) &= 0 \\
h''''(t) &= 0,\ h^{(5)}(t) = 0,\ \text{etc.}
\end{aligned}
$$

Rule: If $p(x)$ is a polynomial and n is greater than the degree of $p(x)$, then $p^{(n)}(x) = 0$.

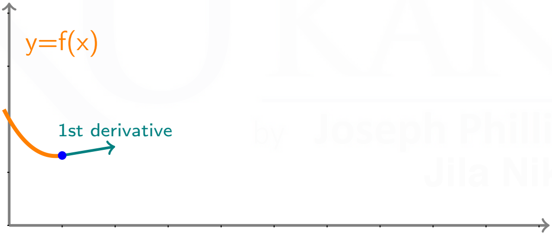
Example: Evaluate

$$\frac{d^{2021}}{dx^{2021}}\left(x^{2005} - 86x^{2001} - 33x^{1984} - 10x^{1066} + 1974x^{105} - 9378\right).$$

Direction of Change

If $f'(x) > 0$ on an interval, then f is **increasing** on that interval.

If $f'(x) < 0$ on an interval, then f is **decreasing** on that interval.



Let $f(x)$ be a differentiable function and let (a, b) be an interval in the domain of f.

Concavity

$f(x)$ is **concave up** on (a, b) if the graph of f lies **above** the tangent line at every point in (a, b).

$f(x)$ is **concave down** on (a, b) if the graph of f lies **below** the tangent line at every point in (a, b).

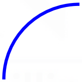
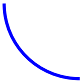
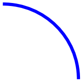
Concavity and the Second Derivative

If $f''(x) > 0$ on (a, b), then f is concave **up** on (a, b).

If $f''(x) < 0$ on (a, b), then f is concave **down** on (a, b).

Direction AND Concavity

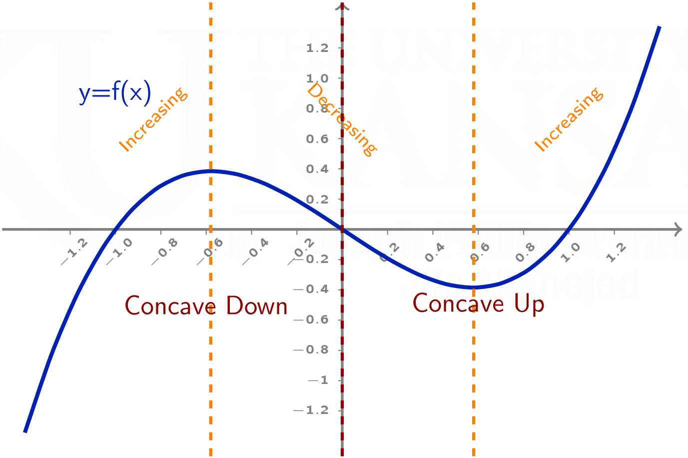
Knowing direction (increasing/decreasing) and concavity (up/down) tells us that the curve has one of four basic shapes.

	Concave up	**Concave down**
Increasing		
Decreasing		

Direction and Concavity

Example I: Let $f(x) = x^3 - x$.

1. On what intervals is f increasing/decreasing? $f'(x) = 3x^2 - 1$
2. On what intervals is f concave up/down? $f''(x) = 6x$



Direction and Concavity

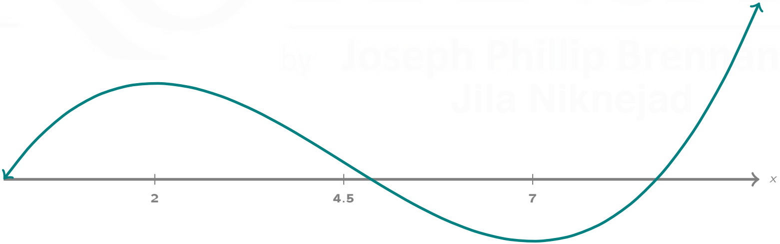
Example II: Sketch the graph of a continuous function f satisfying the following:

(i) $f'(x) > 0$ on the intervals $(-\infty, 2)$ and $(7, \infty)$.

(ii) $f'(x) < 0$ on the interval $(2, 7)$.

(iii) $f''(x) < 0$ on the interval $(-\infty, 4.5)$.

(iv) $f''(x) > 0$ on the interval $(4.5, \infty)$.



Direction and Concavity

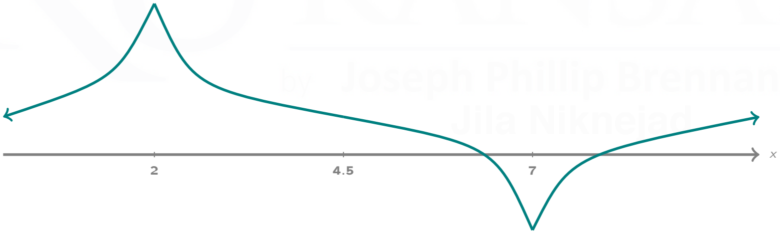
Example III: Sketch the graph of a continuous function f satisfying the following:

(i) $f'(x) > 0$ on the intervals $(-\infty, 2)$ and $(7, \infty)$.

(ii) $f'(x) < 0$ on the interval $(2, 7)$.

(iii) $f''(x) > 0$ on the interval $(-\infty, 4.5)$.

(iv) $f''(x) < 0$ on the interval $(4.5, \infty)$.



An inspection of the graph reveals that f' is undefined at $x = 2$ and at $x = 7$. So f'' is also undefined at those values.

Factorials Review

$n!$ is the number $n(n-1)(n-2)\cdots(3)(2)(1)$, thus

$$\begin{aligned}
1! &= 1\\
2! &= 2\\
3! &= 6\\
4! &= 24\\
5! &= 120\\
n! &= n(n-1)! = n(n-1)(n-2)! = \ldots
\end{aligned}$$

By convention $0! = 1$.

Example IV: Calculate the first four derivatives of $y = x^{-1}$. Then find the pattern and determine a general formula for $y^{(n)}$.

$$y^{(0)} = x^{-1}$$

$$y^{(1)} = (-1)x^{-2} = -x^{-2}$$

$$y^{(2)} = (-1)(-2)x^{-3} = 2x^{-3}$$

$$y^{(3)} = (-1)(-2)(-3)x^{-4} = -6x^{-4}$$

$$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5}$$

$$y^{(n)} = (-1)^n n! x^{-(n+1)}$$

Higher Derivatives and Implicit Differentiation

Example V: Find y'' for the ellipse defined implicitly by

$$4x^2 + y^2 = 9.$$

Implicitly differentiating with respect to x yields

$$8x + 2yy' = 0. \qquad (*)$$

which we can solve for y' to get

$$y' = \frac{-4x}{y}. \qquad (**)$$

Differentiate $(*)$ again:

$$8 + 2(yy'' + y'y') = 0.$$

Solve for y'' and then plug in $(**)$:

$$y'' = \frac{-4 - y'y'}{y} = \frac{-4 - \left(\frac{-4x}{y}\right)^2}{y} = \frac{-4y^2 - 16x^2}{y^3}$$

Higher Derivatives and Implicit Differentiation

Example V: (continued)

$$4x^2 + y^2 = 9 \qquad y' = \frac{-4x}{y} \qquad y'' = \frac{-4y^2 - 16x^2}{y^3}$$

Sign of y':
positive if x, y have different signs, otherwise negative

Sign of y'':
opposite of sign of y